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Analytical redefinition of DOL and managerial investment decisions

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Abstract

Purpose – The purpose of this paper is to develop alternative analytical measures for the degree of operating leverage (*DOL*) that reflect the impact of uncertain demand shocks in the product's market on optimal production levels, sales and profits of the firm.

Design/methodology/approach – The elasticity measures are constructed according to a theoretical formulation of optimal production level that corresponds to demand shocks for given predetermined levels of fixed cost.

Findings – The paper suggests two main findings. First, the analytical marginal *DOL* is at least twice the traditional *DOL* depending on the structure of the shock, the production function and demand's elasticity. The traditional *DOL* is equal to the measure only when large-scale negative demand prompts the firm to abandon production. Second, the paper also provides an analytical measure of *DOL* in terms of elasticity of profit to sales rather than to production level. Both theoretically and empirically elasticity of profit to sales can be better measured and better reflects risk.

Research limitations/implications – This paper should be extended to encompass multiple shocks on demand and supply while investigating the empirical multi variants distribution of the shocks.

Practical implications – The model can be used by managers who are well informed about the fixed and variable costs of their firm. The model determines the mean profit- risk trade off which is an important factor in all investment decision problems.

Originality/value – Surprisingly and according to the best knowledge, this paper is the first attempt in the literature for alternative analytical *DOL*s' formulations that is coherent with basic economic theories of optimal production level under risk.

Keywords Real options, Financial leverage, Operating leverage, DOL, Expected operating profit-risk efficiency analysis, Outsourcing, Elasticity of demand

Paper type Research paper

I. Introduction

Production costs may be divided into variable costs, which vary with production levels, and fixed costs, which do not. A general classification of a specific type of costs as variable or fixed may not exist. For example, some labor costs can be fixed and others can be variable, depending on production and operating profitability levels and timing durations or the magnitude of fluctuations in the demand for and supply of labor. Corporate finance textbooks (see e.g. Brealy *et al.*, 2006, pp. 225-226) correctly claim that cost structures with higher fixed costs and lower variable costs are associated with greater risk. All else being equal, an organization with a higher proportion of

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fixed costs relative to variable costs is said to have a higher degree of operating leverage (*DOL*). Most financial and managerial textbooks define *DOL* as the elasticity of operating profit (EBIT) to changes in sales or level of production (see Damodaran, 2001, pp. 202-203). Unfortunately, these examples assume a constant average price per unit and an average variable cost per unit at all levels of production or sales (Brigham and Ehrhardt, 2005, pp. 552-553; and Brealy *et al.* 2006, pp. 277-278). Based on these two assumptions, the analytical measure of elasticity of operating profit with respect to production or sales volume is simply one plus the ratio of fixed costs to operating profits. This analytical measure of *DOL* is very popular and widely used, especially in managerial economics textbooks (see definitions in, Ross *et al.*, 2000, pp. 335-338; Salvatore, 2004; Hirschey, 2003). However, because formal financial reports of firms do not distinguish between fixed and variable expenses, empirical studies examining the effect of operating leverage on risk premiums have considered various proxies for these values. These proxies include estimates using time series regressions in which the dependent variable is the periodic change in operating profits and the independent variable is the change in sales. In addition, various studies have correctly claimed that the analytical equation for *DOL* of one plus the ratio of fixed costs to operating profits is a poor risk measure because it approaches positive or negative infinity as the firm approaches the breakeven point (Dran, 1991; Hodgin and Kiymaz, 2005). Other studies have correctly noted that this *DOL* measure ignores the real option of changing the level of production to its optimal level when there is a change in the prices of inputs and outputs (see e.g. Booth, 1991; Enajero, 2012). Despite the various definitions and difficulties of estimating operating leverage, it is widely accepted that operating leverage is positively associated with risk premiums and high book-to-market ratios (see e.g. Ferri and Jones, 1979; Novy-Marx, 2011; Feijoo and Jorgensen, 2010).

This paper does not address the empirical relationship between operating leverage and risk. Instead, we begin with a model that defines the optimal production behavior in the face of uncertain supply and demand given the real option to determine the level of production. This model facilitates the development of more general analytical expressions of *DOL* which is generally contrasted to the traditional measures of the textbooks' *DOL*s.

The next section provides some preliminaries with a short literature review. The model and new analytical method of calculating the *DOL* and examples are provided in all Sections III up to VIII. Section IX presents some practical managerial applications, while Section X which is the last section, concludes the paper.

II. Preliminaries and scientific background

The *DOL* can be measured analytically in many ways. For example, *DOL* can be measured as the ratio of fixed costs to variable costs or by the ratio of fixed assets to current assets. The widely accepted general definition of the *DOL* is the elasticity of operating profit π with respect to sales or level of production Q. In other words:

$$DOL = \frac{d\pi}{dQ} \cdot \frac{Q}{\pi} \tag{1}$$

where Q is the number of physical units, and π is defined by:

$$\pi = (p - v)Q - FC \tag{2}$$

where p is price per unit, v is variable cost per unit and FC is the fixed cost.

Lev's (1974) seminal study is most likely the first study to examine the effect of operating leverage on risk. Unfortunately, Lev claimed that "In an uncertain environment, future demand, Q_{jt} (in physical units), is of course a random variable" (p. 629) rather than specifying that the product's price is a random variable. Q is determined according to the endogenous optimal production level given the revealed ex-post level of the ex-ante random price. Accordingly, Lev claimed that $d\pi/dQ = (p - v)$. Based on this claim, it is easy to conclude that the analytical equation for *DOL* is simply:

$$DOL = (p - v) \cdot \frac{Q}{\pi} = \frac{sales - variable\ expenses}{profit} = 1 + \frac{FC}{\pi} \quad (3)$$

Thus, Lev concluded that a higher *DOL* is associated with a higher "contribution margin" ($p-v$). Furthermore, given an assumed price p in an industry, Lev concluded that a higher *DOL* is associated with a lower variable cost per unit v. Then, Lev proved that the intra-industry level of β is negatively associated with v, implying that the intra-industry *DOL* is positively associated with the β measure of risk[1]. Because formal financial reports of companies do not report their variable costs separately from their fixed costs, various studies that followed Lev's study used different empirical methods for estimating *DOL*. These empirical studies also examined the relative effect of operating leverage and financial leverage on the risk premium of equity[2],[3]. Our paper develops analytical expressions for alternative *DOL*s. Managers who have a complete knowledge about a company's production functions and have no difficulty in analytically calculating *DOL* can use these definitions and the principles of our optimization processes. Accordingly, this paper does not present a full review of the vast array of empirical studies that are related to operating leverage.

The expression of *DOL* in Equation (3) is used extensively in textbooks on corporate finance and managerial economics. This paper will first show that Equation (3) is incorrect when either the prices of outputs or inputs or both are exogenous random variables and the level of production (or sales) is the endogenous decision variable. Based on this approach, our paper will construct analytical expressions for the elasticity of operating profit with respect to price and sales.

It should be noted that many previous studies have considered the effects of uncertain exogenous demand and supply shocks and the structure of assets and liabilities on the risk premium and the value of the corporation. For example, Shrieves (1981) assumed price uncertainty and presented the efficient frontier between expected operating profit and risk under the CAPM in terms of the covariance between operating profit and market return. In his model, Shrieves derived an optimal ratio between fixed costs (capital) and variable costs (labor) but did not generate an analytical term for *DOL*. Dotan and Ravid (1985) developed a one period valuation model of the firm in which the price of the product is a random variable, $\tilde{P} = P + \tilde{u}$, and the optimal capacity (investment) is determined to maximize the value of the firm. They considered the optimal capacity level to be a function of the *DOL*. In addition, they presented only the implicit first-order condition for the optimal capacity with and without debt financing. Their main interest was the comparative statics of this solution, not an analytical *DOL*. They found that the optimal level of capacity decreases with increasing levels of debt.

Our managerial approach differs from that of Dotan and Ravid (1985) in that we are interested in establishing an analytical definition of *DOL*. We also want to determine the tradeoff curve between operating profits and an appropriate alternative analytical

DOL risk measures, the latter being determined by both the cost structure of the firm and the type of risk that the firm faces.

Booth (1991) developed a one period valuation model that is based on state preference contingent claim optimization. Booth defined *DOL*s differently than in Equation (3). His first *DOL* is based on the assumption that there is no real option for changing the level of future production and that all costs are fixed (Equation (20) on p. 120). His second *DOL* is based on the assumption that there is a real option for changing the level of future production (Equation (21) on p. 120). Booth's *DOL*s are relevant only under conditions of perfect competition. Their use is impractical because an estimation of state preference utility factors is required to calculate these *DOL*s.

Various other studies have criticized the analytical expression of *DOL* in Equation (3). Dran, 1991 correctly claimed that *DOL* in Equation (3) is simply:

$$DOL = \frac{q}{q-1} \tag{4}$$

where q is the breakeven level of production that results in zero operating profits. Thus, Drain claimed that *DOL* is related only to the breakeven point q and not "necessarily to the relationship between fixed and variable costs" (see there; p. 89). However, it is very clear that the breakeven point q is determined by fixed and variable costs and price per unit. Thus, Equation (4) does not negate the impact of variable and fixed costs on *DOL*, especially when the price per unit is known. Hodgin and Kiymaz (2005) continued this critique of Drain by claiming that as the level of production approaches the breakeven point q, *DOL* approaches positive infinity from above and negative infinity from below. Thus, *DOL* cannot serve as a reliable risk factor.

In the next sections, we present a different approach in which managers select production levels and efficient combinations of fixed and variable costs to minimize risk for any level of operating profit. The results of our model are new analytical expressions of *DOL*.

III. The model

Let's start with a simplified basic model of a single product firm (activity)[4] operating in one period. At the beginning of the period, the firm decides on the level of fixed costs *FC* (due to investment as well as other fixed expenses not necessarily related to investment). The demand function $\tilde{p}(Q)$ for the product is random due to a random exogenous shock $\tilde{\delta}$ that is revealed at the end of the period. The firm has the real option of changing the production level Q at the end of the period according to the revealed demand function. If the ex-post δ is very negative and the loss is above the fixed cost *FC* at the optimal production level, the firm abandons the project and pays only the fixed costs. Once the production level is determined, the variable costs and product price are determined. According to this model, the beginning of the period's ex-ante random operating profit $\tilde{\pi}$ that will be realized at the end of the period is:

$$\tilde{\pi} = (1+\tilde{\delta})p_0(Q)Q - c(Q,FC)Q - FC \tag{5}$$

where $\tilde{\delta}$ is the random end-of-period shock to the expected demand function $p_0(Q)$. The variable cost per unit $c(Q,FC)$ is a function of the level of production and the level of *FC*. It is also assumed that $p'_0(Q) \leqslant 0$ and $p''_0(Q) \geqslant 0$. The case of $p'_0(Q) = 0$ is the case of perfect competition in the product's market. For simplicity, we also assume that

$c(Q, FC) = c(Q) + b(FC)$. In other words, the effect of fixed costs on variable costs is independent of Q. We also assume that $c'(Q) \geqslant 0$, $c''(Q) \leqslant 0$, $b'(FC) \leqslant 0$ and $b''(FC) \geqslant 0$.

At the end of the period, δ is revealed, and the manager determines the optimal level of production Q^* that maximizes the end-of- period operating profit π. At the beginning of the period, the managers select the efficient level of FC, where efficiency is determined based on the tradeoff between the expected operating profit and the expected risk. The level of risk in this paper will be measured by the analytical expression for the expected elasticity of the operating profit with respect to production, price or sales level. We also assume that at the end of the period, the firm has a real option to change the production levels to maximize the expected operating profit[5].

We believe that the assumption of $p_0(Q)(1+\tilde{\delta})$ is more reasonable than $p_0(Q) + \tilde{\delta}$ though some previous studies analyzed the risk while assuming $p_0(Q) + \tilde{\delta}$ (see e.g. Dotan and Ravid 1985; Booth 1991). In order to better understand why the assumption $p_0(Q)(1+\tilde{\delta})$ is more reasonable, recall that $\delta = 1$ USD is a significant shock at the bottom lower right of the demand curve and a much smaller shock at the upper left end of the demand curve. $p_0(Q)(1+\tilde{\delta})$ is equivalent to an assumption of a percentage change in the price for any given quantity.

According to our basic assumptions, the ex-post operating profit π for any selected level of FC and a revealed δ is:

$$\pi = (1+\delta)p_0(Q)Q - (c(Q) + b(FC))Q - FC \tag{6}$$

The first-order condition for selecting Q^* that maximizes the operating profit π is:

$$Q^* = \frac{(1+\delta)p_0(Q^*) - c(Q^*) - b(FC)}{c'(Q^*) - (1+\delta)p_0'(Q^*)} \tag{7}$$

Q^* in Equation (7) can be solved explicitly once the functions $p(Q)$ and $c(Q)$ are specified. Even from the implicit form in Equation (7) one can conclude that Q^* increases with p, p' and δ and decreases with c, c' and b.

The maximum realized operating profit at the optimal production level is:

$$\pi^* = [(1+\delta)p_0(Q^*) - (c(Q^*) + b(FC))]Q^* - FC \tag{8}$$

By inserting Equation (7) into Equation (8), we obtain an expression for the optimal production level:

$$\pi^* = (Q^*)^2[c'(Q^*) - (1+\delta)p_0'(Q^*)] - FC \tag{9}$$

Equation (9) will be used in the development of the *DOL* terms below.

IV. The risk

Let us now consider the risk in terms of the marginal (point) elasticity of operating profits at the optimal production level with respect to the optimal production level Q^*, the price at the optimal production level and sales at the optimal production level. We denote these elasticity terms as $DOL(Q^*)$, $DOL(P(Q^*))$ and $DOL(S(Q^*))$. We start with $DOL(Q^*)$ and compare it to the traditional DOL, which is denoted as $DOL(Q)$.

$DOL(Q^*)$ is defined by the elasticity of operating profit at the optimal production level with respect to production level due to shock $d\delta$ to the demand function for the product. In other words:

$$DOL(Q^*) \equiv \frac{d\pi(Q^*) = \frac{\partial \pi(Q^*)}{\partial \delta} d\delta}{dQ^* = \frac{\partial Q^*}{\partial \delta} d\delta} \cdot \frac{Q^*}{\pi(Q^*)} = \frac{\frac{\partial \pi(Q^*)}{\partial \delta}}{\frac{\partial Q^*}{\partial \delta}} \cdot \frac{Q^*}{\pi(Q^*)} \tag{10}$$

Note that if we use the conventional textbook $DOL(Q)$ in Equation (3), at the optimal production level, $DOL \equiv \partial \pi(Q^*)/\partial Q^* \cdot Q^*/\pi(Q^*)$. This traditional DOL ignores the effect of the change in the demand function because the derivative assumes that the price is constant, and only the level of production has changed. However, there is no economic reason for a change in the level of production when there is no change in prices. Thus, the assumptions behind the traditional DOL contradict basic economic principles.

Theorem 1. Given the assumptions of our model, $DOL(Q^*)$ is:

$$DOL(Q^*) = \frac{1}{\pi^*}\left\{2(\pi^* + FC) - \frac{(Q^*)^3 p_0'(Q^*)[2B^2 + A(c''(Q^*) - (1+\delta)p_0''(Q^*)]}{[B \cdot p_0(Q^*) + A p_0'(Q^*)]}\right\} \tag{11}$$

where:

$$A = (1+\delta)p_0(Q^*) - c(Q^*) - b(FC) \tag{12}$$

is the "operating profit gross margin," which is the sales minus the variable expenses per unit that is also given in the numerator of Equation (7) and:

$$B = c'(Q^*) - (1+\delta)p_0'(Q^*) \tag{13}$$

is the denominator of Equation (7).

The proof is presented in Appendix 1.

The $DOL(Q^*)$ in Equation (11) is a result of the existence of the real option of changing the production level at the end of the period to maximize operating profits based on the revealed price. A comparison of the statistics of the $DOL(Q^*)$ in Equation (11) reveals that $DOL(Q^*)$ increases with FC and $-p'_0(Q^*)$ and decreases with π^*, Q^* and $p_0(Q^*)$.

In the case of perfect competition, $p'_0(Q) = 0$ and the right-hand side expression of the numerator in Equation (11) vanishes, giving us the following corollary:

Corollary 1. In the case of a perfect competitive product market, $DOL(Q^*)$ is:

$$DOL(Q^*) = 2(1 + \frac{FC}{\pi^*}) = 2\frac{\textit{sales} - \textit{variable expenses}}{\textit{operating profit}} \tag{14}$$

The result in Equation (14) is quite striking because the "correct" DOL is twice the textbooks' DOL. The simple intuition behind this result is that the textbooks' DOL considers only changes in Q. According to our model, a shock to prices leads to changes in both Q^* and $P(Q^*)$, and the two changes are in the same direction. Note that in the case of a demand function with a negative slope, $p'_0(Q) < 0$, and the whole expression to the right of the left-hand side of Equation (11) is positive. In other words, in an imperfect competitive situation $DOL(Q^*) > 2(1 + FC/\pi^*)$.

Theorem 2. Under perfect competition or in case $\tilde{p}(Q) = p_0(Q) + \tilde{\delta}$;

$$DOL(Q^*) = 2(1 + \frac{FC}{\pi^*}) = 2\frac{sales - variable\ expenses}{operating\ profit} \quad (15)$$

The proof is presented in Appendix 2.

The difference between *Theorems 1* and *2* indicates that the *DOL* as a measure of risk is sensitive to the way in which the random shock operates on the demand curve. When δ, the risk factor, is not infinitely small, the two alternative assumptions $\tilde{p}(Q) = p_0(Q)(1 + \tilde{\delta})$ and $\tilde{p}(Q) = p(Q) + \tilde{\delta}$ can lead to a large difference in the estimation of *DOL*.

V. *DOL*(*P*(*Q**)) analysis – the elasticity of operating profit to price

We define the elasticity of operating profit to changes in price as:

$$DOL(P(Q^*)) \equiv \frac{d\pi(Q^*) = \frac{\partial \pi^*}{\partial \delta} d\delta}{dp(Q^*) = \frac{\partial p(Q^*)}{\partial \delta} d\delta} \cdot \frac{p(Q^*)}{\pi(Q^*)} = \frac{\frac{\partial \pi^*}{\partial \delta}}{\frac{\partial p(Q^*)}{\partial \delta}} \cdot \frac{(1+\delta)P_0(Q^*)}{\pi(Q^*)} \quad (16)$$

Theorem 3:

$$DOL(P(Q^*)) = \frac{\frac{2[\pi(Q^*)+FC]}{Q^*} \cdot \frac{\partial Q^*}{\partial \delta} - (Q^*)^2 p'_0(Q^*)}{P_0(Q^*) + (1+\delta)P'_0(Q^*)\frac{\partial (Q^*)}{\partial \delta}} \cdot \frac{(1+\delta)P_0(Q^*)}{\pi(Q^*)} \quad (17)$$

and:

$$\frac{\partial Q^*}{\partial \delta} = \frac{B \cdot p(Q^*) + Ap'(Q^*)}{2B^2 + A(c''(Q^*) - (1+\delta)p''(Q^*))} \quad (18)$$

where A and B are given in Equations (12) and (13).

The definition of $DOL(P(Q^*))$ is presented in Appendix 3.

$DOL(P(Q^*))$ increases with FC, P_0 and δ decreases with π.

In the case of a perfectly competitive market, $p'(Q^*) = 0$. Therefore, at the optimal production level:

$$DOL((P(Q^*)) = \frac{[(1+\delta) \cdot p_0(Q^*) - Ac''(Q^*)]Q^*}{\pi(Q^*)} = \frac{Sales - c''(Q^*) \cdot variable\ \cos t}{operating\ profit} \quad (19)$$

If we also assume (as in the next example) that $c''(Q^*) = 0$, then at the optimal production level:

$$DOL((P(Q^*)) = \frac{(1+\delta) \cdot p_0(Q^*) \cdot Q^*}{\pi(Q^*)} = \frac{sales(Q^*)}{operating\ profit(Q^*)} \quad (20)$$

If we apply instead the traditional approach and calculate the elasticity of operating profit to the product's price while holding the production level constant, we also obtain the same ratio of:

$$DOL(P(Q)) = \frac{P_0(1+\delta)}{\pi(Q_0)} = \frac{sales}{operating\ profit} \tag{21}$$

However, Equation (20) holds only at the optimal production level Q^*, whereas according to the traditional approach, Equation (21) holds at any level of production.

VI. *DOL*(*S*(*Q**)) analysis – the elasticity of operating profit to sales

It is likely that the analytical equation for the elasticity of operating profit to sales is more reasonable and more practical than the elasticity of operating profit with respect to either production levels or prices. We make this assertion because the value of sales is the product of quantity and price, so it reflects both price and production level. Thus, managers may consider the sensitivity of operating profit to changes in sales as a better measure of risk than the sensitivity of operating profit only to either price or quantity alone. In addition, financial reports provide data about sales, but information about price and quantity is more difficult to obtain. Thus, empirical studies of *DOL* generally examine the relationship between operating profit and sales. Therefore, $DOL(S(Q^*))$ measures the elasticity of operating profit to sales when the optimal production level Q^* is assumed. The definition of $DOL(S(Q^*))$ is:

$$DOL(S(Q^*)) \equiv \frac{\frac{\partial \pi^*}{\partial \delta}}{\frac{\partial (Q^* \cdot P(Q^*))}{\partial \delta}} \cdot \frac{Q^* P(Q^*)}{\pi(Q^*)} \tag{22}$$

Theorem 4. The marginal elasticity of operating profit to sales is given by:

$$DOL(S(Q^*)) = \frac{2\left[1 + \frac{FC}{\pi(Q^*)}\right] \cdot \frac{\partial Q^*}{\partial \delta} - \frac{(Q^*)^3 p_0'(Q^*)}{\pi(Q^*)}}{\frac{Q^*}{(1+\delta)} + \frac{\partial Q^*}{\partial \delta}\left[1 + \frac{Q^* \cdot p_0'(Q^*)}{P_0(Q^*)}\right]} \tag{23}$$

where:

$$\frac{\partial Q^*}{\partial \delta} = \frac{B \cdot p_0(Q^*) + A p_0'(Q^*)}{2B^2 + A[c''(Q^*) - (1+\delta) p_0''(Q^*)]} \tag{24}$$

and A and B are given in Equations (12) (13).

We do not provide a full proof of *Theorem 4* as one can obtain the terms (23) and (24) by a simple mathematical use of Equations (9) and (2-A).

Let us now consider (23) and (24) under the case of a perfectly competitive market where $\partial P_0(Q^*)/\partial Q^* = 0$ and thus Equation (23) is reduced to:

$$DOL(S(Q^*)) = \frac{\partial \pi^*}{\partial \delta} \cdot \frac{1}{\pi(Q^*)\left[\frac{1}{(1+\delta)} + \frac{\partial Q^*}{\partial \delta}\frac{1}{Q^*}\right]} \tag{25}$$

Recall from Equations (9) and (2-A) that $\partial\pi^*/\partial\delta = 2[\pi(Q^*) + FC]/Q \cdot \partial Q^*/\partial\delta - (Q^*)^2 p_0'(Q^*)$. We substitute this result in Equation (25) and rearrange the terms to obtain the following result for a perfectly competitive market:

$$DOL(S(Q^*)) = \frac{2\left[1 + \frac{FC}{\pi(Q^*)}\right] \cdot \frac{\partial Q^*}{\partial \delta}}{\frac{Q^*}{(1+\delta)} + \frac{\partial Q^*}{\partial \delta}} \tag{26}$$

Corollary 2. Under perfect competition:

I. $DOL(S(Q^*)) < DOL(Q^*)$.
II. and sufficient conditions for $DOL(S(Q^*)) < DOL(Q^*) < DOL(P(Q^*))$ are:

$$\frac{Variable\ \cos t}{Sales} > \frac{1}{2} \tag{27}$$

and $c''(Q^*) = 0$.

The proof of part I of the corollary is immediately evident by comparing Equations (26) and (14). The proof of the second part of the corollary results from Equations (20) and (22). The lower elasticity of the operating profit to sales relative to the elasticity to price is not surprising. For example, when there is a negative shock to the price, the firm reduces the quantity of the product to restore the optimal level of production that leads to the maximum operating profit given the new prices.

The use of the various equations for *DOL* requires estimates of demand and production. It is clear that those estimates can vary among firms. Thus, the example below, which is based on very simplified assumptions, may not be relevant for firms with other demand and production structures.

VII. A numerical example

Assume the following simplifying assumptions:

$$p(Q) = p_0 - p_1 \cdot Q \rightarrow p'(Q) = -p_1 \tag{A}$$

$$c(Q) = c_0 + c_1 \cdot Q \rightarrow c'(Q) = c_1 \tag{B}$$

$$b(FC) = \frac{b}{FC} \tag{C}$$

Assume also the following parameters:

$p_O = 12$, $\delta = -0.25$, $c_0 = 4$, $b = 80$, $FC = 40$, $c_1 = 0.03$, $p_1 = 0.02$

p_O, p_1, c_O, c_1 and b are all positive terms. Note that positive p_1 and c_1 are equivalent to linearly decreasing demand functions and linearly increasing variable expenses per unit functions, respectively. The total variable cost function is thus $C(Q) = c_0 \cdot Q + c_1 \cdot Q^2$. Hence, the marginal cost function increases with production levels according to $C'(Q) = c_0 + 2c_1 \cdot Q$. An increasing marginal variable cost function is required to obtain an increasing supply function.

After rearranging terms, Q^* in Equation (7) is solved explicitly as:

$$Q^* = \frac{(1+\delta)p_0 - c_0 - \frac{b}{FC}}{2(c_1 + (1+\delta)p_1)} = \frac{0.75 \cdot 12 - 4 - 80/40}{2(0.03 + 0.75 \cdot 0.02)} = \frac{3}{0.09} \equiv \frac{A}{B} = 33.333 \tag{7$'$}$$

and $\pi(Q^*)$ in Equation (9) is solved explicitly as:

$$\pi(Q^*) = [c_1 + (1+\delta)p_1] \cdot (Q^*)^2 - FC = [0.03 + 0.75 \cdot 0.02] \cdot (33.3333)^2 - 40 = 10 \qquad (9)'$$

According to Appendix 1, $DOL(Q^*)$ can also be written as:

$$DOL(Q^*) = \frac{2(\pi^* + FC) - \frac{(Q^*)^3 p'(Q^*)}{\frac{\partial Q^*}{\partial \delta}}}{\pi^*} \qquad (11)'$$

and:

$$\frac{\partial Q^*}{\partial \delta} = \frac{B \cdot p(Q^*) + Ap'(Q^*)}{2B^2 + A(c''(Q^*) - (1+\delta)p''(Q^*))}$$

A and B are given in Equations (11) and (12), respectively. In our example, $A = 1.5$ and $B = 0.045$.

Also recall that in our example $p(Q) = p_0 - p_1 \cdot Q \rightarrow p'(Q) = -p_1 = -0.02$, $c(Q) = c_0 + c_1 \cdot Q \rightarrow c'(Q) = c_1 = 0.03$ and $b(FC) = b/FC = 80/40 = 2$ to conclude that:

$$\frac{\partial Q^*}{\partial \delta} = \frac{B^* p(Q^*) + Ap'(Q^*)}{2B^2 + A[c''(Q^*) + (1+\delta)p''(Q^*)]} = \frac{B^*(p_0 - p_1 Q) - Ap_1}{2B^2} = \frac{0.45 * (12 - 0.02 \cdot 33.333) - 1.5 \cdot 0.02}{2 \cdot 0.45^2} = 118.519$$

Thus, in our example:

$$DOL(Q^*) = \frac{2(\pi^* + FC) - \frac{(Q^*)^3 p'(Q^*)}{\frac{\partial Q^*}{\partial \delta}}}{\pi^*} = \frac{2(10+40) + \frac{(33.333)^3 0.02}{118.519}}{10} = 10.6246$$

The traditional $DOL(Q)$ is only 5.

Now let's calculate $DOL(P(Q^*))$ from Equation (17).

$$DOL(P(Q^*)) = \frac{\frac{2[\pi(Q^*)+FC]}{Q^*} \cdot \frac{\partial Q^*}{\partial \delta} - (Q^*)^2 p_0'(Q^*)}{P(Q^*) + (1+\delta)P_0'(Q^*)\frac{\partial (Q^*)}{\partial \delta}} \cdot \frac{(1+\delta)P_0(Q^*)}{\pi(Q^*)}$$
$$= \frac{\frac{100}{33.3333} \cdot 118.519 + (33.3333)^2 0.02}{(12 - 33.333 \cdot 0.02) - 0.75 \cdot 0.02 \cdot 118.519} \cdot \frac{0.75 \cdot (12 - 33.333 \cdot 0.02)}{10}$$
$$= 33.605$$

In our example, the operating profit is much more sensitive to changes in price than to changes in quantity. According to part two of *Corollary 2*, $DOL(P(Q^*)) > DOL(Q^*)$ whenever variable costs are more than 50 percent of sales. In our example, the price per unit is $0.75(12-0.02 \times 33.3333) = 8.5$, and the cost per unit is $4 + 0.03 \times 33.3333 + 80/40 = 7$. In other words, the variable costs are 82.3 percent of sales, far above 50 percent of sales. Thus, $DOL(P(Q^*)) = 33.605 > DOL(Q^*) = 10.625$.

Let us calculate $DOL(S(Q^*))$, which is the elasticity of operating profit with respect to sales. From Equation (22):

$$DOL(S(Q^*)) = \frac{2\left[1+\frac{FC}{\pi(Q^*)}\right]\cdot\frac{\partial Q^*}{\partial\delta} - \frac{(Q^*)^3 p_0'(Q^*)}{\pi(Q^*)}}{\left[\frac{Q}{(1+\delta)} + \frac{\partial Q^*}{\partial\delta}\left(1+p_0'(Q^*)\frac{Q}{P_0(Q^*)}\right)\right]}$$

$$= \frac{2[1+\frac{40}{10}]\cdot 118.519 + \frac{(33.333)^3 0.02}{10}}{\left[\frac{33.3333}{(0.75)} + 118.519\left(1-0.02\frac{33.333}{12-0.02\times 33.333}\right)\right]} = 8.073$$

VIII. Large-scale risk

Recall that $DOL(Q^*)$, $DOL(P(Q^*))$ and $DOL(S(Q^*))$ measure the elasticity of the operating profit to changes in optimal production levels, price and sales, respectively. Infinite changes in the demand curve lead to infinite changes in the optimal production level. However, when large shocks occur, the results are different. Let's define $\Delta\delta \equiv \delta_1 - \delta_0$ where δ_0 is the δ before the shock to the demand curve. Let us also define $DOL_L(Q^*)$, $DOL_L(P(Q^*))$ and $DOL_L(S(Q^*))$ due to large changes in δ as:

$$DOL_L(Q^*) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{Q^*_{\delta_1} - Q^*_{\delta_0}}\cdot\frac{Q^*_{\delta_0}}{\pi(Q^*_{\delta_0})} \tag{28}$$

$$DOL_L(P(Q^*)) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{P(Q^*_{\delta_1}) - P(Q^*_{\delta_0})}\cdot\frac{P(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})} \tag{29}$$

and:

$$DOL_L(S(Q^*)) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{S(Q^*_{\delta_1}) - S(Q^*_{\delta_0})}\cdot\frac{S(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})} \tag{30}$$

where $\pi(Q^*_{\delta_0})$, $P(Q^*_{\delta_0})$ and $Q^*_{\delta_0}$ are the operating profit given the price of the product, the price of the product and the optimal production level at an assumed δ_0. $\pi(Q^*_{\delta_1})$, $P(Q^*_{\delta_1})$ and $Q^*_{\delta_1}$ are the operating profit given the price, the price of the produce and optimal production level due to $\delta_1 \equiv \Delta\delta + \delta_0$.

Theorem 5.

I. Above the abandonment point $DOL_L(Q^*)$, $DOL_L(P(Q^*))$ and $DOL_L(S(Q^*))$ increase with $\Delta\delta$ and:

II. When $\Delta\delta \equiv \delta_1 - \delta_0 < 0$ and very negative and the firm approach the abandonment level from above, in which case the loss approaches the fixed cost, and the optimal production level approaches zero, then $DOL_L(Q^*) = DOL_L(S(Q^*) = DOL(Q)$. In other words:

$$DOL_L(Q^*) = DOL_L(S(Q^*)) = DOL(Q) \equiv 1 + \frac{FC}{\pi_{\delta_0}} \tag{31}$$

and:

$$DOL_L(P(Q^*)) = \frac{P(Q^*_{\delta_0})}{P(Q^*_{\delta_0}) - P(Q^*_{\delta_1})} \tag{32}$$

III. When $\Delta\delta$ leads to zero operating profit, the DOL_L are:

$$DOL_L(Q^*) \equiv \frac{Q^*_{\delta_0}}{Q^*_{\delta_0} - Q^*_{\delta_1}} \tag{33}$$

$$DOL_L(P(Q^*)) \equiv \frac{P(Q^*_{\delta_0})}{P(Q^*_{\delta_0}) - P(Q^*_{\delta_1})} \tag{34}$$

and:

$$DOL_L(S(Q^*)) = \frac{S(Q^*_{\delta_0})}{S(Q^*_{\delta_0}) - S(Q^*_{\delta_1})} \tag{35}$$

Proof

Part I of the Theorem in which we claim that $DOL_L(Q^*)$, $DOL_L(P(Q^*))$ and $DOL_L(S(Q^*))$ increase with $\Delta\delta$ is simply due to the convexity nature of the elasticity terms. This result is due to the existence of the real option to change the production level according to $\Delta\delta$. The real option generates the convexity of the DOL_L terms. In other words, the more negative $\Delta\delta$ is, the more the firms will reduce their production levels to slow the decline in profits. In contrast, when $\Delta\delta$ starts to increase, the firms will increase their production levels and the operating profit increases at an accelerated rate. The convexity phenomenon is well known in all option models. This feature is well reflected in Table I. There we can see that above the abandonment point all the large elasticity terms strictly increase with $\Delta\delta$.

	$\delta = -0.25$							$\delta = 0.0$					
	DOL_L			Levels of				DOL_L			Levels of		
$\Delta\delta$	Q^*	P^*	S^*	P^*	Q^*	π^*	$\Delta\delta$	Q^*	P^*	S^*	P^*	Q^*	π^*
0.4	19.63	55.84	11.05	12.1	73.58	247	0.4	4.18	7.34	2.22	14.2	93.1	462.8
0.35	18.5	53.31	10.75	11.7	69.23	209.2	0.35	4.02	7.14	2.19	13.8	89.5	416.3
0.3	17.38	50.71	10.43	11.2	64.71	173.5	0.3	3.86	6.94	2.15	13.4	85.7	371.4
0.25	16.25	48.04	10.1	10.8	60	140	0.25	3.7	6.74	2.11	13	81.8	328.2
0.2	15.13	45.31	9.75	10.4	55.1	108.8	0.2	3.54	6.53	2.08	12.5	77.8	286.7
0.15	14	42.5	9.37	9.9	50	80	0.15	3.38	6.31	2.04	12.1	73.6	247
0.1	12.88	39.62	8.97	9.44	44.68	53.8	0.1	3.21	6.09	2	11.7	69.2	209.2
0.05	11.75	36.65	8.54	8.97	39.13	30.4	0.05	3.05	5.86	1.96	11.2	64.7	173.5
0	10.63	33.6	8.07	8.5	33.33	10	0	2.89	5.63	1.91	10.8	60	140
−0.05	9.5	30.47	7.57	8.02	27.27	−7.3	−0.05	2.73	5.39	1.86	10.4	55.1	108.8
−0.1	8.38	27.25	7.02	7.53	20.93	−21.2	−0.1	2.57	5.14	1.82	9.9	50	80
0.15	7.25	23.93	6.42	7.03	14.29	−31.4	0.15	2.41	4.89	1.76	9.44	44.7	53.8
−0.2	6.13	20.52	5.75	6.52	7.32	−37.8	−0.2	2.25	4.63	1.71	8.97	39.1	30.4
−0.25	5	17	5	6	0	−40	−0.25	2.09	4.36	1.65	8.5	33.3	10
							−0.3	1.93	4.08	1.59	8.02	27.3	−7.3
							−0.35	1.77	3.8	1.52	7.53	20.9	−21.2
							−0.4	1.61	3.51	1.45	7.03	14.3	−31.4
							−0.5	1.29	2.89	1.29	6	0	−40

Table I. $\Delta\delta$ and $DOL_L(Q^*)$, $DOL_L(P(Q^*))$ and $DOL_L(S(Q^*))$

Part II of the *Theorem 5* is based on the assumption that at the abandonment point production level and sales are zero and profit is –FC. Thus, Equation (31) is obtained as follows:

$$DOL_L(Q^*) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{Q^*_{\delta_1} - Q^*_{\delta_0}} \cdot \frac{Q^*_{\delta_0}}{\pi(Q^*_{\delta_0})} = \frac{-FC - \pi(Q^*_{\delta_0})}{0 - Q^*_{\delta_0}} \cdot \frac{Q^*_{\delta_0}}{\pi(Q^*_{\delta_0})} = 1 + \frac{FC}{\pi(Q^*_{\delta_0})} = DOL(Q)$$

and also:

$$DOL_L(S(Q^*)) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{S(Q^*_{\delta_1}) - S(Q^*_{\delta_0})} \cdot \frac{S(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})} = \frac{-FC - \pi(Q^*_{\delta_0})}{0 - S(Q^*_{\delta_0})} \cdot \frac{S(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})} = 1 + \frac{FC}{\pi(Q^*_{\delta_0})} = DOL(Q)$$

Equation (32) is obtained as follows:

$$DOL_L(P(Q^*)) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{P(Q^*_{\delta_1}) - P(Q^*_{\delta_0})} \cdot \frac{P(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})} = \frac{-FC - \pi(Q^*_{\delta_0})}{P(Q^*_{\delta_1}) - P(Q^*_{\delta_0})} \cdot \frac{P(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})}$$
$$= DOL(Q)\frac{P(Q^*_{\delta_0})}{P(Q^*_{\delta_1}) - P(Q^*_{\delta_0})}$$

Part III of *Theorem 5* is obtained simply by plugging $\pi(Q^*_{\delta_1}) = 0$ into Equations (33)-(35) to obtain the following specific cases.

The specific case of Equation (33):

$$DOL_L(Q^*) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{Q^*_{\delta_1} - Q^*_{\delta_0}} \cdot \frac{Q^*_{\delta_0}}{\pi(Q^*_{\delta_0})} = \frac{0 - \pi(Q^*_{\delta_0})}{Q^*_{\delta_1} - Q^*_{\delta_0}} \cdot \frac{Q^*_{\delta_0}}{\pi(Q^*_{\delta_0})} = \frac{Q^*_{\delta_0}}{Q^*_{\delta_0} - Q^*_{\delta_1}}$$

The specific case of Equation (34) is:

$$DOL_L(P(Q^*)) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{P(Q^*_{\delta_1}) - P(Q^*_{\delta_0})} \cdot \frac{P(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})} = \frac{0 - \pi(Q^*_{\delta_0})}{P(Q^*_{\delta_1}) - P(Q^*_{\delta_0})} \cdot \frac{P(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})}$$
$$= \frac{P(Q^*_{\delta_0})}{P(Q^*_{\delta_0}) - P(Q^*_{\delta_1})}$$

And the specific case of Equation (35) is:

$$DOL_L(S(Q^*)) \equiv \frac{\pi(Q^*_{\delta_1}) - \pi(Q^*_{\delta_0})}{S(Q^*_{\delta_1}) - S(Q^*_{\delta_0})} \cdot \frac{S(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})} = \frac{0 - \pi(Q^*_{\delta_0})}{S(Q^*_{\delta_1}) - S(Q^*_{\delta_0})} \cdot \frac{S(Q^*_{\delta_0})}{\pi(Q^*_{\delta_0})}$$
$$= \frac{S(Q^*_{\delta_0})}{S(Q^*_{\delta_0}) - S(Q^*_{\delta_1})}$$

Table I presents the numerical exposition of $DOL_L(Q^*)$, $DOL_L(P(Q^*))$ and $DOL_L(S(Q^*))$. The table presents two cases. In both cases, all of the parameters besides δ are equivalent to those in the previous numerical example. In the first case, which is presented on the right-hand side of the Table I, the initial δ is $\delta_0 = -0.25$, as in the previous numerical example. In the second case, $\delta_0 = 0$ as expected according to our model at time zero. Given the higher demand function, the second case is a more profitable situation than the first. Thus, the $DOL_L(Q^*)$, $DOL_L(P(Q^*))$ and $DOL_L(S(Q^*))$ terms in the second case are less than their equivalents in the first case.

The decision to abandon the product occurs in case I when $\Delta\delta = -0.25$ and in case II when $\Delta\delta = -0.50$. Thus, $DOL_L(Q^*) = DOL_L S(Q^*) = 5$ in the first case, and $DOL_L(Q^*) = DOL_L S(Q^*) = 1.29$ in the second case as claimed in part II of *Theorem 5*. In both cases, these large risk DOL_Ls are exactly the same as the traditional marginal $DOL(Q)$ for $\Delta\delta = 0$.

IX. Practical managerial applications

Given the level of investment (fixed costs), the manager's decision variable is the optimal production level under different shocks to demand function. Let first examine this optimal production decision according to the example in Section VII. In addition, below we will extend the example in Section VII to consider the impact of additional investment on the expected profit and risk.

To be reminded, the parameters and results of the numerical example in Section VII are: Expected price after, $P_0(1 + \delta) = 12 \times (1-0.25) = 9$.

The variable costs per production unit Q is:

$$c(Q, FC) = c(Q) + b(FC) = c_0 + c_1 \cdot Q + \frac{b}{FC} = 4 + 0.03Q + \frac{80}{40}$$

as $c_0 = 40$, $c_1 = 0.03$ and $b = 80$ and $FC = 40$.

The demand curve per unit Q is: $p(Q) = p_0 - p_1 Q = 12 - 0.02Q$

The results are: optimal output $Q^* = 33.333$; The implied price per unit $P^* = 11.333$; The implied operating profit $\pi(Q^*) = 10$; The elasticity of operating profit to production level, $DOL(Q^*) = 10.6246$ (where the traditional textbook's $DOL(Q)$ is only 5); The elasticity of operating profit to price per unit, $DOL(P(Q^*)) = 33.333$; and the elasticity of operating profit to sales, $DOL(S(Q^*)) = 8.073$.

The above results can assist the manager with the following important projections: a given marginal change in the demand curve generates a 10.6246 times higher percentage change in the profits than in the percentage in the optimal production level. The same demand shock leads to a percentage change in profit which is 33.333 times higher than the percentage change in price and 8.073 times higher than the percentage change in sales. Namely, a given marginal change in the demand curve is expected to lead to a change in the optimal output which is 33.333/10.624 = 3.14 times higher than the change in price. According to Table I in Section VIII, $\Delta\delta = -5$ percent reduce profits from 10 to a loss of -7.3 and P^*decreases from 8.50 to 8.02 and Q^*decreases from 27.7 to 20.93. The percentage change in the optimal production relative to the percentage change in price is: $((33.33 - 27.27)/33.33)/((8.50 - 8.02)/8.50 = 3.21)$.

It is worth noting that the firm's sales managers that are compensated on the basis of volume of deals naturally tend to prefer discount in price rather than losing deals and clients. However, according to our example the general manager should direct the operating manager to reduce production level from 27.7 to 20.93 (-24.4 percent) due to the 5 percent reduction in demand. That reduction is 3.21 times higher than the expected decrease in price. By that direction the manager maximizes the profit.

A second important application is related to the classical capital budgeting risk. Assume the manager examines an increase in the investment level which determines the level of fixed costs. Also assume for simplicity the each additional investment step is confined to a size of ten. Meaning, the manager would like to examine increases from the current investment level of 40 (see the example in Section VII) to levels of 50, 60, 70 or even 80. The main findings are presented in Table II.

An increase of the investment (*FC*) from 40 to 50 leads to an increase in the optimal production level from 33.33 to 37.78 (+13.3 percent). The operating profit increases from 10.00 to 14.2 (+42.0 percent). The risk in terms $DOL(Q^*)$ decreases from 10.60 to 9.67 (−9.4 percent), $DOL(P(Q^*))$ decreases from 33.6 to 26.52 (−21.1 percent) and $DOL(S(Q^*))$ decreases from 8.07 to 7.09 (−12.1 percent). Thus, an increase of the investment from 40 to 50 should be executed as expected profit increases and risk decreases due to this additional investment.

When the investment is increased from 50 to 60 the expected profit slightly increased from 14.20 to 14.70 (+3.5 percent) but the risk in terms of $DOL(Q^*)$, $DOL(P(Q^*))$ and $DOL(S(Q^*))$ goes up by 13.1, 3.7 and 10.57 percent, respectively. Thus, the decision to increase the investment from 50 to 60 depends on the risk attitude of the manager.

The increase of the investment from 60 to 70 lower expected profit from 14.7 to 12.70 (−13.6 percent) and also increases the risk in terms of $DOL(Q^*)$, $DOL(P(Q^*))$ and $DOL(S(Q^*))$ by 29.0, 32.4 and 26.8 percent, respectively. It is clear that the increase of the investment from 60 to 70 should not be considered.

If the investment is increased from 70 to 80 then the results becomes much worse. Profits goes down from 12.70 to 8.90 (−29.90 percent) and the risk in terms of $DOL(Q^*)$, $DOL(P(Q^*))$ and $DOL(S(Q^*))$ goes up by 53.57, 47.11 and 51.7 percent, respectively.

Note that if the sizes of the investment are not confined to steps of tens, then by a simple application of the Solver application of EXCEL one can found in our example that the maximum attainable expected profit of 14.9 is attained for investment size of 56.43.

Minimum $DOL(Q^*)$ of 9.62 is obtained when investment size is 47.7.

Minimum $DOL(P(Q^*))$ of 26.31 is obtained when investment size is 52.8.

And minimum $DOL(S(Q^*))$ of 7.08 is obtained f when investment size is 49.1.

Thus, if for example $DOL(S(Q^*))$ is the relevant risk measure, than the manager should consider according to his risk preferences only investment's sizes in the range of 49.1-56.43, as the lowest point of the range leads to minimum $DOL(S(Q^*))$ and top of the range leads to the highest expected profit.

Once the manager determines the specific parameters of demand and supply (p_0, p_1,c_0 and c_1) and the relationship between fixed costs and variable costs (in our case b/FC), then the computations can be easily executed by the equations in our model. Different production functions will change the basic calculations. However, the main ideas of the paper can easily be used to adapt any specific production functions.

Table II.
The impact of investment (fixed cost-FC) on expected profit and risk

FC	Q^*	Operating profit	$DOL(Q^*)$	$DOL(P(Q^*))$	$DOL(S(Q^*))$
40	33.33	10.00	10.60	33.60	8.07
50	37.78	14.20	9.67	26.52	7.09
60	40.74	14.70	10.94	27.52	7.84
70	42.85	12.70	14.11	33.45	9.94
80	44.44	8.90	21.67	49.21	15.08

X. Summary and conclusion

This paper suggests alternative analytical measures to the well-known textbook definition of the *DOL*, which can be useful from a managerial point of view. To the best of our knowledge, this is a first theoretical attempt to model *DOL* with respect to both large and small uncertain demand shocks. We demonstrate the impact of the uncertain shocks to the demand function on optimal production level, sales and profits. Our one period model assumes that fixed costs are determined at time zero and that there is a real option to change optimal production at time one according to the revealed demand function. Based on that model, we find that the marginal analytical *DOL* is at least twice higher than the well-known analytical *DOL* in the literature. In the case of perfect competition, our analytical marginal *DOL* is exactly twice the traditional marginal *DOL*. Our large-scale *DOL* is equal to the traditional analytical marginal *DOL* only when a large-scale negative demand allows the firm to reach the point of abandonment of the production process. The suggested measures of *DOL* can be applied practically to better estimate the expected impact of demand shock on product price and production level. In addition, the model can help the managers to select levels of investment (fixed costs) according to their impact on profit and risk. Future research should extend this work to consider also the impact of uncertain shocks of the supply function and the implication of correlations between shocks to the demand and supply functions.

Notes

1. Lev supported his theoretical claim by an empirical study. The main difficulty in Lev's empirical study was estimating v, which is not provided in companies' formal financial reports.
2. See for example Gahlon (1980), Gahlon and Gentry (1982), Mandelker and Rhee (1984) and Huffman (1989). Guthrie (2011) claimed that expected return does not necessarily increase monotonously with operating leverage when allowances are made for abandoning unprofitable projects.
3. Novy-Marx (2011) found that DOL could add 35 percent to the explanatory power of the Fama and French three-factor model in cases of intra-industry analysis.
4. If the firm is a multi-product firm, it is assumed that managers are able to disaggregate the firm into separate activities. The overall firm is simply the sum of all of the separate activities.
5. The financial literature considers value maximization rather than operating profit maximization. However, value maximization is just the selection of a specific point on the efficient frontier of expected operating profit and risk. Thus, the mean operating profit-risk efficiency analysis is a relevant managerial analysis.

References

Booth, L. (1991), "The influence of production technology on risk and cost of capital", *Journal of Financial and Quantitative Analysis*, Vol. 26, pp. 109-127.

Brealy, A.R., Myers, S.C. and Allen, F. (2006), *Corporate Finance, 8th International*, McGraw-Hill Irwin, New York, NY.

Brigham, E.F. and Ehrhardt, M.C. (2005), *Financial Management*, 11th ed., South Western Thomson, Mason, IA.

Damodaran, A. (2001), *Corporate Finance*, 2nd ed., John Wiley & Sons Inc., New York, NY.

Dotan, A. and Ravid, S.A. (1985), "On the interaction of real and financial decisions of the firm under uncertainty", *Journal of Finance*, Vol. 40 No. 5, pp. 501-517.

Dran, J. (1991), "A different perspective on operating leverage", *Journal of Economics and Finance*, Vol. 15, Spring, pp. 87-95.

Enajero, S.E. (2012), "Curriculum integration: does the degree of operating leverage defy optimal input combinations", *Journal of Higher Education Theory and Practice*, Vol. 12 No. 3, pp. 43-49.

Feijoo, L.G. and Jorgensen, R.D. (2010), "Can operating leverage be the cause of the value premium", *Financial Management*, Vol. 39 No. 3, pp. 1127-1157.

Ferri, M.G. and Jones, W.H. (1979), "Determinants of financial structure: a new methodological approach", *Journal of Finance*, Vol. 34 No. 3, pp. 631-634.

Gahlon, M. (1980), "Operating leverage as a determinant of systematic risk", *Journal of Business Research*, Vol. 9, September, pp. 297-308.

Gahlon, J.M. and Gentry, J.A. (1982), "On the relationship between systematic risk and the degree of operating leverage", *Financial Management*, Vol. 11 No. 1, pp. 15-23.

Guthrie, G. (2011), "A note on operating leverage and expected rates of returns", *Finance Research Letters*, Vol. 8 No. 2, pp. 88-100.

Hirschey, M. (2003), *Managerial Economics*, 10th ed., South-Western Publishing, Mason, IA.

Hodgin, K. and Kiymaz, H. (2005), "Extending the operating profit elasticity of operating leverage in managerial economics texts", *Journal of Economics and Economics Educations*, Vol. 6 No. 3, pp. 23-36.

Huffman, S.P. (1989), "The effect of the degrees of operating leverage and financial leverage on the systematic risk of common stock", *Quarterly Journal of Business and Economics*, Vol. 28 No. 1, pp. 83-100.

Lev, B. (1974), "On the association between operating leverage and risk", *The Journal of Financial and Quantitative Analysis*, Vol. 9 No. 4, pp. 627-641.

Mandelker, G.N. and Rhee, S.G. (1984), "The impact of the degrees of operating and financial leverage on systematic risk of common stock", *The Journal of Financial and Quantitative Analysis*, Vol. 19 No. 1, pp. 45-57.

Novy-Marx, R. (2011), "Operating leverage", *Review of Finance*, Vol. 15 pp. 103-134.

Ross, S.A., Westerfield, R.W. and Jordan, B.D. (2000), *Fundamentals of Corporate Finance*, 5th ed., Irwin McGraw-Hill, New York, NY.

Salvatore, D. (2004), *Managerial Economics in a Global Economy*, 5th ed., South-Western Publishing, Mason, IA.

Shrieves, R.E. (1981), "Uncertainty, the theory of production, and optimal operating leverage", *Southern Economic Journal*, Vol. 47 No. 3, pp. 690-702.

Appendix 1. Proof of DOL(Q*) in Equation (11)

The definition of *DOL* (*Q**) in Equation (10) is:

$$DOL(Q^*) \equiv \frac{\frac{\partial \pi(Q^*)}{\partial \delta}}{\frac{\partial Q^*}{\partial \delta}} \cdot \frac{Q^*}{\pi(Q^*)} \tag{10}$$

According to Equation (9) $\pi(Q^*) = (Q^*)^2[c'(Q^*) - (1+\delta)p_0'(Q^*)]FC$ thus:

$$\frac{\partial \pi(Q^*)}{\partial \delta} = [c'(Q^*) - (1+\delta)p_0'(Q^*)]2Q^* \cdot \frac{\partial Q^*}{\partial \delta} + (Q^*)^2\left[\frac{\partial c'(Q^*)}{\partial \delta} \cdot \frac{\partial Q^*}{\partial \delta} - p_0'(Q^*) - (1+\delta)\frac{\partial p_0'(Q^*)}{\partial \delta} \cdot \frac{\partial Q^*}{\partial \delta}\right] \tag{A1}$$

However, under the very reasonable assumption that the slopes of the average price and costs are not affected by the finite shock δ, it can be assumed naturally that $\partial c'(Q^*)/\partial\delta = \partial p_0'(Q^*)/\partial\delta = 0$ can we, therefore rewrite Equation (1-A) as:

$$\frac{\partial\pi(Q^*)}{\partial\delta} = [c'(Q^*) - (1+\delta)p_0'(Q^*)]2Q^* \cdot \frac{\partial Q^*}{\partial\delta} - (Q^*)^2 p_0'(Q^*) \quad \text{(A2)}$$

Thus based on Equation (9) and above derivatives and assumptions we obtain:

$$DOL(Q^*) \equiv \frac{2(\pi(Q^*) + FC) - \frac{(Q^*)^3 p_0'(Q^*)}{\frac{\partial Q^*}{\partial\delta}}}{\pi(Q^*)} \quad \text{(A3)}$$

Based on the solution of the optimal production level $Q^* = ((1+\delta)p_0(Q^*) - c(Q^*) - b(FC))/(c'(Q^*) - (1+\delta)p_0'(Q^*) \equiv (A/B))$ in Equation (7), after rearranging and cancelling terms, we find that: $\partial Q^*/\partial\delta$ is:

$$\frac{\partial Q^*}{\partial\delta} \cdot B^2 = B \cdot \left(p_0(Q^*) - B\frac{\partial Q^*}{\partial\delta}\right) - A\left[c''(Q^*)\frac{\partial Q^*}{\partial\delta} - p_0'(Q^*) - (1+\delta)p_0''(Q^*)\frac{\partial Q^*}{\partial\delta}\right] \quad \text{(A4)}$$

Equation (4-A) can also be written as:

$$\frac{\partial Q^*}{\partial\delta} = \frac{B \cdot p_0(Q^*) + Ap_0'(Q^*)}{2B^2 + A[c''(Q^*) - (1+\delta)p_0''(Q^*)]} \quad \text{(A5)}$$

thus:

$$DOL(Q^*) = \frac{1}{\pi^*}\left\{2(\pi^* + FC) - \frac{(Q^*)^3 p_0'(Q^*)[2B^2 + A(c''(Q^*) - (1+\delta)p_0''(Q^*)]}{(B \cdot p_0(Q^*) + Ap_0'(Q^*))}\right\} \quad \text{(A6)}$$

■

Appendix 2. Proof of Theorem 2

When $\tilde{p}(Q) = p_0(Q) + \tilde{\delta}$, the non-random operating profit at the end of the period is:

$$\pi = (p_0(Q) + \delta)Q - (c(Q) + b(FC))Q - FC \quad \text{(A7)}$$

and the first-order condition for Q^*that maximizes π is:

$$Q^* = \frac{p_0(Q^*) + \delta - c(Q^*) - b(FC)}{c'(Q^*) - p_0'(Q^*)} \quad \text{(A8)}$$

Thus, the maximum operating profit π^*can be written in terms of Q^* as:

$$\pi^* = (c'(Q^*) - p_0'(Q^*))(Q^*)^2 - FC \quad \text{(A9)}$$

By definition:

$$DOL(Q^*) = \frac{\frac{\partial\pi^*}{\partial\delta}}{\frac{\partial Q^*}{\partial\delta}} \cdot \frac{Q^*}{\pi^*} \quad \text{(A10)}$$

Let's calculate $(\partial \pi^*)/(\partial \delta)$ and $(\partial Q^*)/(\partial \delta)$ in order to find $DOL(Q^*)$:

$$\frac{\partial \pi^*}{\partial \delta} = [c'(Q^*) - p'_0(Q^*)]2Q^* \cdot \frac{\partial Q^*}{\partial \delta} + (Q^*)^2 \left[\frac{\partial c'(Q^*)}{\partial \delta} \cdot \frac{\partial Q^*}{\partial \delta} - \frac{\partial p'_0(Q^*)}{\partial \delta} \cdot \frac{\partial Q^*}{\partial \delta}\right] \tag{A11}$$

Once again, let's assume that $(\partial c'(Q^*))/(\partial \delta) = (\partial p'_0(Q^*))/(\partial \delta)$to rewrite Equation (5-B) as:

$$\frac{\partial \pi^*}{\partial \delta} = [c'(Q^*) - p'_0(Q^*)]2Q^* \cdot \frac{\partial Q^*}{\partial \delta} \tag{A12}$$

By plugging Equation (6-B) into Equation (4-B) and using Equation (3-B), we find that:

$$DOL(Q^*) = 2\left(1 + \frac{FC}{\pi^*}\right) \tag{A13}$$

when $\tilde{p}(Q) = p_0(Q) + \tilde{\delta}$.

■

Appendix 3. Proof of Theorem 3
By definition:

$$DOL(P*) = \frac{\frac{\partial \pi^*}{\partial \delta}}{\frac{\partial p(Q^*)}{\partial \delta}} \cdot \frac{(1+\delta)P_0(Q^*)}{\pi(Q^*)} \tag{A14}$$

According to Appendix 1:

$$\frac{\partial \pi^*}{\partial \delta} = [c'(Q^*) - (1+\delta)p'_0(Q^*)]2Q^* \cdot \frac{\partial Q^*}{\partial \delta} - (Q^*)^2 p'_0(Q^*) \tag{A15}$$

by definition $p(Q^*) = (1+\delta)P_0(Q^*)$.
Thus:

$$\frac{\partial p(Q^*)}{\partial \delta} = P_0(Q^*) + (1+\delta)P'_0(Q^*)\frac{\partial (Q^*)}{\partial \delta} \tag{A16}$$

thus:

$$DOL(P*) = \frac{[c'(Q^*) - (1+\delta)p'(Q^*)]2Q^* \cdot \frac{\partial Q^*}{\partial \delta} - (Q^*)^2 p'(Q^*)}{P_0(Q^*) + (1+\delta)P'_0(Q^*)\frac{\partial (Q^*)}{\partial \delta}} \cdot \frac{(1+\delta)P_0(Q^*)}{\pi(Q^*)} \tag{A17}$$

Based on Equation (9), $\pi^* = (Q^*)^2[c'(Q^*) - (1+\delta)p'_0(Q^*)]FC$. Thus, Equation (4-D) can be written as:

$$DOL(P*) = \frac{\frac{2[\pi(Q^*)+FC]}{Q^*} \cdot \frac{\partial Q^*}{\partial \delta} - (Q^*)^2 p'(Q^*)}{P_0(Q^*) + (1+\delta)P'_0(Q^*)\frac{\partial (Q^*)}{\partial \delta}} \cdot \frac{(1+\delta)P_0(Q^*)}{\pi(Q^*)} \tag{A18}$$

Based on Equation (3-A) in Appendix 1 $\partial Q^*/\partial \delta = (B \cdot p_o(Q^*) + Ap'_0(Q^*))/(2B^2 + A[c''(Q^*) - (1+\delta)\, p''_0(Q^*)])$
where $Q^* = ((1+\delta)p_0(Q^*) - c(Q^*) - b(FC))/(c'(Q^*) - (1+\delta)p'_0(Q^*) \equiv (A/B))$

For the case when $p'(Q^*) = 0$, Equation (4-C) is reduced to Equation (6-C).

$$DOL(P*) = \frac{c'(Q^*)2Q^* \cdot \frac{\partial Q^*}{\partial \delta} \cdot (1+\delta)}{\pi(Q^*)} \tag{A19}$$

But also:

$$Q^* = \frac{(1+\delta)p_0(Q^*) - c(Q^*) - b(FC)}{c'(Q^*)} \tag{A20}$$

and thus:

$$\frac{\partial Q^*}{\partial \delta} = \frac{c'(Q^*)p_0(Q^*) - Ac''(Q^*)}{2(c'(Q^*))^2} \tag{A21}$$

Thus,

$$DOL(P*) = \frac{(1+\delta)[p_0(Q^*) - Ac''(Q^*)]Q^*}{\pi(Q^*)} \tag{A22}$$

If we also assume $c''(Q^*) = 0$ (as in our example), then:

$$DOL(P*) = \frac{(1+\delta)p_0(Q^*)Q^*}{\pi(Q^*)} = \frac{Sales}{Profit} \tag{A23}$$

Part 3 of Appendix 3**:** finding $FC_P{}^{**}$ that minimizes $DOL(P(Q^*)$

Under the simplifying assumptions of the example plus the assumption of $P_1 = 0$, the expected $DOL(P(Q^*)$ according to Equation (20) is simply:

$$E(DOL(P(Q^*)) = \frac{p_0 \cdot E(Q^*)}{\pi(E(Q^*))} \tag{A24}$$

Recall that under the simplifying assumptions, $E(Q^*) = (p_0 - c_0 - (b/FC))/(2c_1)$ and $\pi(E(Q^*)) = c_1 \cdot (E(Q^*))^2 - FC$, allowing us to conclude that:

$$E(DOL(P(Q^*)) = \frac{p_0 \cdot \frac{p_0 - c_0 - \frac{b}{FC}}{2c_1}}{c_1 \cdot \left(\frac{p_0 - c_0 - \frac{b}{FC}}{2c_1}\right)^2 - FC} \tag{A25}$$

rearranging the terms to obtain:

$$E(DOL(P(Q^*)) == \frac{p_0 \cdot \left(p_0 - c_0 - \frac{b}{FC}\right)}{0.5 \cdot \left(p_0 - c_0 - \frac{b}{FC}\right)^2 - 2c_1 \cdot FC} \tag{A26}$$

The derivative of Equation (16-D) with respect to FC yields:

$$\frac{\partial E(DOL(P(Q^*))}{\partial FC} = \frac{\frac{P_0 \cdot b}{FC^2}[0.5A^2 - 2c_1 \cdot FC] - P_0A\left[A\frac{b}{FC^2} - 2c_1\right]}{\left[0.5 \cdot \left(p_0 - c_0 - \frac{b}{FC}\right)^2 - 2c_1 \cdot FC\right]^2} \tag{A27}$$

where A $= (p_0 - c_0 - (b/FC))$

$FC_P{}^{**}$ that equate to zero the numerator of (17-D) is:

$$-0.5A^2\frac{P_0 \cdot b}{FC^2} - 2c_1 \cdot \frac{P_0 \cdot b}{FC} + 2c_1 \cdot P_0A = 0 \tag{A28}$$

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